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Temperley–Lieb algebra, group theory and the Potts model

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Abstract. We show that a scheme based on the critical Andrews-Baxter-Forrester model may be used to construct all known representations of operators obeying the Temperley-Lieb relations. Using associated groups we show, for a small number of operators, that this scheme gives the complete set of irreducible representations. We obtain the degeneracies of the transfer matrix spectrum for the q-state Potts model.

In a recent letter (Martin 1987a, hereafter referred to as I) we showed how to write down irreducible representations of the operators $\{U_i, i=1,\ldots,2n-1\}$ satisfying the Temperley-Lieb relations (Temperley and Lieb 1971):

$$\begin{aligned} &U_i^2 = q^{1/2} U_i \\ &U_i U_{i\pm1} U_i = U_i \\ &U_i U_j = U_j U_i \qquad |i-j|>1. \end{aligned} \tag{1}$$

The square lattice q-state Potts model n-site layer transfer matrix (Baxter 1982) may be written in terms of a reducible representation of these operators as follows:

$$T = \left(\prod_{i=1}^{n} (x + U_{2i-1})\right)\left(\prod_{i=1}^{n-1} (1 + xU_{2i})\right) \tag{2}$$

where $x = q^{-1/2}(\exp(\beta)-1)$ and β is the coupling parameter. Several other statistical mechanical models may be written by using different representations of the same operators (see I). For a given model the decomposition into irreducible representations is of particular interest, since it corresponds to a block diagonalisation of the transfer matrix. Different blocks are then responsible for the free energy and the various different correlation functions governing the long-range behaviour of the model. The degeneracies of irreducible representations occurring in the decomposition will in turn give the degeneracies of eigenvalues in the transfer matrix spectrum. The decomposition into irreducible representations is also of particular interest because it picks out the 'Beraha' q values

$$q = 4\cos^2(\pi/r) \qquad r = 3, 4, 5\ldots \tag{3}$$

as special cases. These are cases which may be associated with conformal field theories at the critical point $x_c = 1$ (Kuniba *et al* 1986). We would like to examine this connection.

A first step in this direction is to associate the Temperley-Lieb operators with groups. This also allows us, in principle, to check the completeness of our set of irreducible representations using orthogonality theorems (see later). In the present

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paper we start by reviewing and generalising the procedure for writing down irreducible representations, including several new results. We then discuss possible associated groups (one such is the Artin braid group) and their uses. We check the completeness of our set of representations for a small number of operators, and obtain the degeneracies of the transfer matrix spectrum using group theoretical techniques.

In certain statistical mechanical models the relations (1) can be intimately connected with the star-triangle equation. This equation is a commutativity condition on transfer matrices which, under some further conditions, leads to solvability of the relevant model. We are not concerned at the moment with such results in themselves (a beautiful exposition may be found in Baxter (1982)), but we may use the statistical mechanical models involved to construct representations of the Temperley-Lieb operators, as follows. It is implicit in previous results (Kuniba *et al* 1986, I) that if a solution $w(a, b, c, d; u)$ of the star-triangle equation (Baxter 1982)

$$\sum_c w(a, b, c, d; u) w(d, c, f, g; v) w(c, b, e, f; v-u)$$
$$= \sum_c w(c, e, f, g; u) w(a, b, e, c; v) w(d, a, c, g; v-u) \tag{4a}$$

can be written in the form

$$w(a, b, c, d; u) = \rho(u)\left(\frac{K_{abcd}}{y(u)} + \delta_{a,c}\right) \tag{4b}$$

where ρ and y depend on a continuous variable u and K depends on discretely valued variables, a, b, c and d, and

$$w(a, b, c, d; 0) = \rho(0)\delta_{a,c} \tag{4c}$$

then the $\{\bar{U}_j\}$ defined by

$$(\bar{U}_j)_{s,s'} \equiv y(u)\left\{\delta_{s,s'} - \left[\left(\prod_{i\neq j} \delta_{s_i,s'_i}\right) w(s_j, s_{j+1}, s'_j, s_{j-1}; u)\right](\rho(u))^{-1}\right\} \tag{4d}$$

where s, s' correspond to specific values for sets of $2n+1$ such discrete variables $\{s_i\}$ and $\{s'_i\}$, are u independent and obey

$$\begin{aligned} &\bar{U}_j^2 = \sqrt{q}\ \bar{U}_j \\ &\bar{U}_j\bar{U}_{j\pm1}\bar{U}_j = R\bar{U}_j \\ &\bar{U}_i\bar{U}_j = \bar{U}_j\bar{U}_i \qquad |i-j|>1 \end{aligned} \tag{4e}$$

where

$$\sqrt{q} = y(u) + y(-u)$$

and

$$R = y(u)y(v-u) + y(v)y(-u) - y(v)y(v-u)$$

are automatically independent of u (by (4)).

In I we focused attention on solutions with $R=1$ which are associated with irreducible representations of the $\{U_i\}$. These solutions arise in the critical square lattice $(r-1)$-state Andrews-Baxter-Forrester (ABF) model (Andrews *et al* 1984) with fixed boundary conditions. Specifically, the representations are written in a basis in which the matrix row and column positions correspond to allowed configurations of the ABF lattice variables in a $2n+1$ site diagonal row: $s_1, s_2, \ldots s_i, \ldots, s_{2n+1}$ (in the ABF model $s_i \in \{1, 2, \ldots, (r-1)\}$ and $|s_i - s_{i+1}| = 1$). The matrix elements $(U_{i-1})_{s,s'}$ are then zero unless all but possibly the ith lattice variables in configurations s and s' (s_i and s'_i, respectively) are the same and $s_{i+1} = s_{i-1}$, whereupon

$$(U_{i-1})_{s,s'} = \left[\sin\left(\frac{s_i\pi}{r}\right)\sin\left(\frac{s'_i\pi}{r}\right)\right]^{1/2}\left[\sin\left(\frac{s_{i+1}\pi}{r}\right)\right]^{-1}. \tag{5}$$

We will give explicit examples of such representations later on. We showed in I that it is the boundary conditions, the values of the first and last lattice variable in a diagonal row (s_1, s_{2n+1}), which label the representation. We showed in particular that with $s_1 = 1$ the representation is irreducible provided that the $(r-1)$-state restriction ($s_i \in \{1, 2, \ldots, (r-1)\}$) is observed.

We now note that it is trivial to extend the scheme to any number of operators l. If we have an even number of operators then the $s_1 = 1$ representations, for example, are labelled by the last lattice variable $s_{l+2} = s_{2n+2} = 2m+2$ (cf $s_{l+2} = s_{2n+1} = 2m+1$ for an odd number of operators) with $m = 0, 1, 2, \ldots, n$; the upper limit is determined by the ABF condition

$$|s_i - s_{i+1}| = 1.$$

The $s_1 = 1$ representations are equivalent to those found using the generalised Young tableau construction of Temperley (1986) in which standard tableaux containing the standard tableau $T^{[r-1]}$ (i.e. the numbered tableau of one row of $r-1$ boxes) are deleted. In this construction such a tableau corresponds to a vanishing idempotent (cf Hamermesh 1962). The relevant idempotents may be constructed as follows (from Temperley 1986): denoting the idempotent for $T^{[r-1]}$ by idem$[r-1]$ we have

$$\mathrm{idem}[r-1] = \mathrm{idem}[r-2]\left(1 - \frac{\sinh((r-2)\theta)\, U_{i+r-3}}{\sinh((r-1)\theta)}\right)\mathrm{idem}[r-2] \tag{6}$$

where idem$[1] = 1$ and $\mathrm{e}^{\theta} + \mathrm{e}^{-\theta} = \sqrt{q}$.

For example, with $r = 4$ ($q = 2$) the idempotent

$$1 + \frac{1}{q-1}(U_i U_{i+1} + U_{i+1} U_i - \sqrt{q}(U_i + U_{i+1})) \tag{7}$$

vanishes in these representations (see later).

In general, however, it is easy to find representations for which these idempotents do not vanish. This is the case, for example, if the $(r-1)$-state restriction in our ABF-based construction is relaxed (as it can be by continuity). Then, after regularising any resultant divergences (consider $s_{i+1} = r$ in (5) or see I), the representation either becomes reducible into a copy of the original (state restriction observed) irreducible representation (if this exists, i.e. if $s_{l+2} < r$) and a copy of a new representation for which the idempotent does not vanish in general; or, if $s_{l+2} \geq r$, forms a new representation (see the appendix) with non-vanishing idempotent. Any further reduction to irreducible representations is manifested (given some care) by vanishing matrix elements in the ABF basis. For example the 42-dimensional representation ($s_1 = 1$, $s_{11} = 1$) with $r = 3$ ($q = 1$) has block diagonal form with the single-element subsets of basis configurations {12121212121} and {12345654321} each labelling one-dimensional blocks (consider $r = 3$ in equation (5)). The former is the original $(r-1)$-state restriction observed representation ($U_i = 1$, all i) and the latter has $U_i = 0$ (all i) in which case the idempotent $1 - q^{-1/2} U_i$ (equation (6)) is non-vanishing.

As another example we will show that the 75-dimensional ($s_1 = 1$, $s_{11} = 5$) representation of the same operators reduces to $40 \oplus 34 \oplus 1$. Consider the 35-element subset of basis configurations with at least one lattice variable $s_i = 6$. Now find the first such variable in the list $s_1, s_2, \ldots, S_{11}$ and perform the transformation

$$s_i \rightarrow 12 - s_i$$

on each subsequent variable. This converts the subset to the complete set of basis configurations for ($s_1 = 1$, $s_{11} = 7$), but cancels out of (5). In other words the two 35-element bases give equivalent representations. In the latter basis the single-element subset {12345678987} labels a one-dimensional block. The generalisation of this procedure to other sets of operators is straightforward. We note for later reference that the 40-element subset of ($s_1 = 1$, $s_{11} = 5$) may be mapped to the 40-element subset of ($s_1 = 1$, $s_{11} = 1$) by performing the transformation

$$s_i \rightarrow 6 - s_i$$

on each variable after the first $s_i = 3$. Again this means that the two representations are equivalent.

Note that we find representations with no equivalent in the Young tableau construction. These include, for example, the equivalent of the Burau representation of Artin's braid group (see later and, for example, Birman (1974)) at the Beraha q values. This is the representation ($s_1 = 1$, $s_{l+2} = l$), or $m = n - 1$, regardless of q.

Furthermore, although away from the Beraha q-values representations with $s_1 \neq 1$ are always reducible to copies of the $s_1 = 1$ representations, this is not the case in general at the Beraha values. For example the 'parasitic' representation given in I for $q = 2$ is the indecomposable representation ($s_1 = 2$, $s_5 = 4$) (after appropriate regularisation using $r = 4 + \varepsilon$ in (5)). Again these representations are not found in the Young tableau construction and their $T^{[r-1]}$ idempotents do not vanish. Conversely all known representations may now be constructed using the ABF-based scheme!

As an explicit example illustrating many of the points discussed above consider the five-site diagonal row ABF model with boundaries fixed to $S_1 = 2$ and $S_5 = 4$. The allowed configurations are {21234}, {23234}, {23434}, {23454} so we have

$$U_1 = \begin{pmatrix} \dfrac{\sin \pi/r}{\sin 2\pi/r} & \dfrac{[\sin(\pi/r)\sin(3\pi/r)]^{1/2}}{\sin 2\pi/r} & & \\ \dfrac{[\sin(\pi/r)\sin(3\pi/r)]^{1/2}}{\sin 2\pi/r} & \dfrac{\sin 3\pi/r}{\sin 2\pi/r} & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & & & 0 \\ & \dfrac{\sin 2\pi/r}{\sin 3\pi/r} & \dfrac{[\sin(2\pi/r)\sin(4\pi/r)]^{1/2}}{\sin 3\pi/r} & \\ & \dfrac{[\sin(2\pi/r)\sin(4\pi/r)]^{1/2}}{\sin 3\pi/r} & \dfrac{\sin 4\pi/r}{\sin 3\pi/r} & \\ 0 & & & 0 \end{pmatrix}$$

$$U_3=\begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & \dfrac{\sin 3\pi/r}{\sin 4\pi/r} & \dfrac{[\sin(3\pi/r)\sin(5\pi/r)]^{1/2}}{\sin 4\pi/r} \\ & & \dfrac{[\sin(3\pi/r)\sin(5\pi/r)]^{1/2}}{\sin 4\pi/r} & \dfrac{\sin 5\pi/r}{\sin 4\pi/r} \end{pmatrix}$$

from equation (5).

Relaxing the $(r-1)$-state condition and putting $r=4+\varepsilon$ we obtain, at leading order in ε,

$$U_1\approx\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & & \\ 1 & 1 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}$$

$$U_2\approx\begin{pmatrix} 0 & & & \\ & \sqrt{2} & \left(\dfrac{\varepsilon\pi}{2}\right)^{1/2} & \\ & \left(\dfrac{\varepsilon\pi}{2}\right)^{1/2} & \dfrac{\varepsilon\pi}{2\sqrt{2}} & \\ & & & 0 \end{pmatrix}$$

$$U_3\approx\frac{2\sqrt{2}}{\varepsilon\pi}\begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 1+\dfrac{3\pi\varepsilon}{16} & \mathrm{i}-\dfrac{\mathrm{i}\pi\varepsilon}{16} \\ & & \mathrm{i}-\dfrac{\mathrm{i}\pi\varepsilon}{16} & -1+\dfrac{5\pi\varepsilon}{16} \end{pmatrix}.$$

After some ε-dependent similarity transformations terms in $1/\varepsilon$ cancel out of U_3 (consider the trace and determinant) and we may set $\varepsilon\to 0$ leaving

$$U_1=\sqrt{2}\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}\qquad U_2=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\qquad U_3=\sqrt{2}\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

It is easy to check that these matrices satisfy the relations (1) with $q=2$. Note that $(r-1)$-state condition relaxed representations are not in general Hermitian. The above representation is called 'parasitic' because the top left 2×2 block also forms a representation—the state restriction observed ($S_1=1$, $S_5=1$) representation—but the other diagonal block does not. This type of representation occurs in the decomposition of some statistical mechanical models (for example the Temperley-Lieb (1971) staggered ice model) but not, for example, in the Potts model (see later). From the point of view of long-range properties of associated statistical mechanical models we are interested only in the irreducible content of this indecomposable representation and representations in general (consider the characteristic polynomial of a transfer matrix

(2) in this representation). In this case we have the top left 2×2 block and the two remaining diagonal elements which form separate irreducible representations.

It is interesting to consider the decomposition, given in Jones (1983), of finite-dimensional C^* algebras A_l whose generators satisfy the relations (1) with Beraha q values. This decomposition gives a sum of those complex matrix algebras generated by the set of $(r-1)$-state restriction observed $s_1=1$ irreducible representations alone. In general the other representations discussed above do not generate these matrix algebras (consider the 'parasitic' representation above).

We now turn to consider the possible groups associated with the relations (1). The first of the relations implies that the $\{U_i\}$ are essentially projections: consider

$$e_i = q^{-1/2}U_i \tag{8}$$

having no inverse. However, the operators

$$t_i = 1 - K(q)U_i \tag{9}$$

may be used to form representation of various groups, depending on the form of the scalar function $K(q)$. The relations (1) imply that the inverse of t_i is also generated by U_i

$$t_i^{-1} = 1 - \tilde{K}(q)U_i \tag{10}$$

where

$$\tilde{K} + K = q^{1/2}\tilde{K}K.$$

The first such group to be discussed in general terms was (see, for example, Temperley 1986) the Artin braid group, which may be represented using

$$K(q) = \mathrm{e}^{\theta} \tag{11}$$

(see also Jones 1985, Kauffman 1986a, b). This is the group of possible braidings of $(l+1)$ strings. The group is infinite for any number of strings $(l+1)$. However, Temperley pointed out that for $q=4$ we have $t_i^2=1$ and the representations reduce to those of the permutation group Σ_{l+1}. It will be useful later to consider the presentations of these groups (see, for example, Magnus *et al* 1976)

$$\langle t_1, \ldots, t_l;\ t_it_{i+1}t_i = t_{i+1}t_it_{i+1},\ t_it_j = t_jt_i\ |i-j|>1\rangle \tag{12}$$

for the $(l+1)$ string braid group, and

$$\langle t_1, \ldots, t_l;\ t_i^2, (t_it_{i+1})^3,\ t_it_j = t_jt_i\ |i-j|>1\rangle \tag{13}$$

for Σ_{l+1}.

It is interesting to note that $q=4$ is not the only exceptional case of (11). When

$$\sum_{j=0}^{k-1} (-\mathrm{e}^{2\theta})^j = 0 \tag{14}$$

we have $t_i^k=1$ as an additional relator in (12). In terms of q we have solutions to (14) at

$$q = 4\cos^2\left(\frac{2s+k}{2k}\pi\right) \tag{15}$$

where $s=1,\ldots,k-1$ and $k>1$. We see that branches of the Beraha q values, in general

$$q = 4\cos^2\left(\frac{\pi m}{r}\right) \qquad m=1,2,\ldots,r-1 \tag{16}$$

correspond to points of additional constraint and therefore potential simplification in the braid group representation. It is not surprising from this point of view, therefore, that the representation structure of the $\{U_i\}$ is different at these points. What is perhaps surprising is the order in which the r values in (16) appear as k increases in (15):

$$\{r_k, k=2,3,4\ldots\}=\{\infty,6,4,10,3,14,8,18,5,22,12,26,7,\ldots\}. \tag{17}$$

In general the braid group of more than two strings is complicated, but at certain Beraha q values the additional relators discussed above allow enough simplification to identify manageable groups. Working with these examples we can begin to explain the structure in (17). We can also check that the ABF-based construction finds all irreducible representations for the $\{U_i\}$ associated with three strings.

For $l=1$ (2 strings), of course, we have the group $\mathbf{Z}_k$ with k inequivalent irreducible representations. The trivial representation corresponds to $U_1=0$ (the ABF basis configuration $\{123\}$), and $U_1=q^{1/2}(\{121\})$ gives

$$t_1=-e^{2\theta} \tag{18}$$

which then accounts for the remaining $(k-2)$ representations by transformations which preserve presentation (12) and $t_1^k=1$, but not the relations (1).

For $l=2$ and $k=2$ we know that Σ_3 has six elements in four classes giving two one-dimensional and two two-dimensional representations. The known set of representations of the $\{U_i\}$ are $U_1=U_2=0$ (from the basis configuration $\{1234\}$) and

$$\begin{aligned} U_1&=\begin{pmatrix}\sqrt{q} & \\ & 0\end{pmatrix}\\ U&=\frac{1}{\sqrt{q}}\begin{pmatrix}1 & q-1\\ 1 & q-1\end{pmatrix}\end{aligned} \tag{19}$$

$(q=4)$ from the configurations $\{1212\}$ and $\{1232\}$. Of course the doubling-up of representations for the $\{t_i\}$ comes from the transformation

$$t_i\rightarrow -t_i$$

which preserves presentation (13) but not relations (1). That is, a representation of the $\{U_i\}$ implies a representation of the $\{t_i\}$ but not necessarily vice versa.

For $l=2$ and $k=3$ $(q=3)$ we have a group of 24 elements in seven classes. At first sight this appears more complicated, although we know only the same pair of representations for the $\{U_i\}$ ($U_i=0$ and $q=3$ in (19)). But this time the transformations

$$t_i\rightarrow at_i$$

where $a^3=1$, induce three representations from each, and there is a three-dimensional representation of the $\{t_i\}$ which does not correspond to a representation of the $\{U_i\}$:

$$\begin{aligned} t_1&=\begin{pmatrix}1 & & \\ & a & \\ & & a^2\end{pmatrix}\\ t_2&=\frac{1}{3}\begin{pmatrix}-a^2 & 2a & 2\\ 2a & -1 & 2a^2\\ 2 & 2a^2 & a\end{pmatrix}.\end{aligned} \tag{20}$$

Here $t_i\rightarrow at_i$ is just a similarity transformation. Counting up (i.e. summing the squares of dimensions of distinct irreducible representations, see, e.g., Hamermesh 1962), we find that the ABF representations of the $\{U_i\}$ are thus the complete set again.

For $k=4$ $(q=2)$ we have a rather complicated group. However, a new relation appears

$$t_1^2t_2^2t_1^2t_2^2=t_2^2t_1^2t_2^2t_1^2. \tag{21}$$

This enables us to construct a subgroup using $\bar{t}_i=t_i^2$ as generators. Since $\bar{t}_i$ may be constructed directly using

$$\bar{t}_i=1-\sqrt{2}\,U_i \tag{22}$$

we can equally well use this subgroup to check the representations of the $\{U_i\}$. The class structure implies four one-dimensional and one two-dimensional representations for the $\{\bar{t}_i\}$. For this group (d_4) the $U_i=0$ representation induces a further three using $t_1\to\pm t_1$ and $t_2\to\pm t_2$ separately (the relation (21) has two of each operator on each side), while all these transformations are simply similarity transformations on the two-dimensional representation (equation (19) with $q=2$). In this example we see the second way in which special cases can arise in the group representations at the Beraha q values—by the appearance of new relations like (21).

We will not consider every k value explicitly. The most interesting example among the remainder is $k=6$ $(q=1)$. Again we use a subgroup—generated by $\bar{t}_i=t_i^3$. This is an infinite group obeying $\bar{t}_i^2=1$ only. The class structure is similar to that of the orthogonal group O(2) (see, e.g., Tung 1985), up to the choice of t_1 or t_2 as the reflection, except that this group has $\mathbb{Z}$ (integers under addition) as a normal subgroup. The trivial representation $U_i=0$ gives the one-dimensional representations (using $\bar{t}_i\to-\bar{t}_i$) and the reducible but indecomposable 2×2 representation ($q=1$ in (19)) gives rise to the infinity of irreducible two-dimensional representations of the $\{\bar{t}_1\}$ through the freedom to make an arbitrary similarity transformation on one of $\bar{t}_1$, $\bar{t}_2$, as follows:

$$U_1=\begin{pmatrix}1 & \\ & 0\end{pmatrix} \qquad \bar{t}_1=1-2U_1=\begin{pmatrix}-1 & \\ & 1\end{pmatrix}$$

$$U_2=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} \qquad \bar{t}_2=1-2U_2=\begin{pmatrix}-1 & \\ -2 & 1\end{pmatrix}$$

$$S_\varphi\bar{t}_2S_\varphi^{-1}=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ -\sin\varphi & -\cos\varphi\end{pmatrix} \qquad (0<\varphi<\pi). \tag{23}$$

Again we find that the ABF scheme gives all the irreducible representations of the $\{U_i\}$.

It is intriguing to note the appearance of the orthogonal group in this context, since the scaling limit of the relevant statistical mechanical models should be associated with the two-dimensional conformal group (Friedan *et al* 1983).

Unfortunately for greater numbers of strings the complications of the braid group ultimately defeat this approach. We note in passing, however, that there are a number of important and tractable cases not addressed above. For example the analysis for $q=3$ is finite at $l=3$.

We now turn to another powerful group, represented using

$$K=\tilde{K}=\frac{2}{\sqrt{q}}. \tag{24}$$

This group has presentation

$$\langle t_1,\ldots,t_l;\ t_i^2,\ t_it_j=t_jt_i\ |i-j|>1\rangle. \tag{25}$$

It has representations in common with the subgroup generated by $t_i^{k/2}$ (with t_i the braid group generator defined in (11) and for even k) when $t_i^k=1$. But, of course, additional relators appear in general in these cases (consider $q=2$ above). This observation will be important later.

Away from the Beraha q values we note that we are free to make distinct similarity transformations on each generator in any representation of (25)

$$t_i \to \tilde{t}_i = S_i t_i S_i^{-1} \tag{26}$$

provided these transformations obey

$$S_i t_j = t_j S_i \quad |i-j|>1. \tag{27}$$

If we make $S_{2i-1}=1$ ($i=1,2,\ldots$) then each distinct set of allowed transformations on the remaining generators will produce a new representation (cf (23)). In particular, but subject to the exceptions at the Beraha q values, this freedom is enough to ensure the existence of a representation which obeys

$$(\tilde{t}_i\tilde{t}_{i+1})^3=1 \tag{28}$$

with a corresponding set of similarity transformations $\{S_i(q)\}$. Such a representation is also a representation of the permutation group Σ_{l+1}. Now consider the Potts representation of the $\{U_i\}$ in the Potts basis (see Baxter 1982). It is easy to generalise this from $2n-1$ to l operators. After some work we find that for $l=2$, for instance, a sufficient condition on the similarity transformation S_2 in this representation is

$$\begin{aligned}\left(\sum_{i=1}^{q}(S_2)_{i1}\right)\left(\sum_{i=1}^{q}(S_2^{-1})_{1i}\right)&=\frac{q}{4} \qquad \text{if } q\neq 1\\ &=q \qquad \text{if } q=1.\end{aligned} \tag{29}$$

Because of the cross product structure of the representation it is possible to generalise this for larger l. Thus, provided there are no additional relators (as can occur at the Beraha q values) the Potts representation provides a representation of Σ_{l+1} via (24) and (26). We can therefore use the orthogonality theorem for characters (see, for example, Leech and Newman 1970) to obtain the decomposition into irreducible representations of Σ_{l+1} and hence of the $\{U_i\}$. The orthogonality theorem is

$$\frac{1}{G}\sum_c d_c\chi_r(g_c)\chi_\mu^\dagger(g_c)=n_\mu \tag{30}$$

where G is the order of the group (in our case $(l+1)!$); the sum is over classes (corresponding to possible partitionings of $l+1$ numbers), d_c is the number of elements in a class (the number of ways of arranging $l+1$ numbers in a given partition into cycles), $\chi_r(g_c)$ is the trace of a class representative group element g_c in representation r and n_μ is the number of copies of irreducible representation μ in r. All the quantities on the left may be calculated when r is the Potts representation (see also Martin 1986). For example, for $l=2$, the quantity

$$\chi_r(\tilde{t}_1\tilde{t}_2)=q-4+\frac{4}{q}\left(\sum_{i=1}^{q}(S_2)_{i1}\right)\left(\sum_{i=1}^{q}(S_2^{-1})_{1i}\right)=q-3. \tag{31}$$

More complicated representatives are identified by reference to the possible partitions of $l+1$ numbers and may be calculated using the cross product structure of the representation. In particular

$$\chi_r(\tilde{t}_1\tilde{t}_2\ldots\tilde{t}_l)=\frac{b_{2l+2}(q)}{b_{l+1}(q)} \tag{32}$$

where

$$b_p(q)=\prod_{r=1}^{[(p-1)/2]}\left[q-4\cos^2\left(\frac{\pi r}{p}\right)\right] \tag{32a}$$

and $[p]$. is the integer part of p. The characters of the irreducible representations χ_μ are discussed, for example, in Hamermesh (1962).

We will give some illustrative examples. Firstly, it is convenient to adopt a specific notation for labelling classes. Following Hamermesh we label a class $(\alpha, \beta, \gamma, \ldots)$ if its elements have α 1-cycles, β 2-cycles, γ 3-cycles and so on, thus

$$\alpha+2\beta+3\gamma+\ldots=l+1$$

and the number of elements in the class is

$$\frac{(l+1)!}{\alpha!(2^\beta\beta!)(3^\gamma\gamma!)\ldots}.$$

Our notation for representations restricts attention to tableaux with only one or two rows (see above). If the length of the second row is zero (and restricting attention to $l=2n-1$ for convenience) then $\mu=n$ and we have the trivial representation for which

$$\chi^{[n+\mu,n-\mu]}_{[\alpha,\beta,\gamma\ldots)}=\chi^{[2n]}_{(\alpha,\beta,\gamma\ldots)}=1$$

where we have adopted an explicit (superscript) tableau notation for the representation. If $n-\mu=1$ then we have

$$\chi^{[2n-1,1]}_{(\alpha,\beta,\gamma\ldots)}=\alpha-1$$

and is a simple diagrammatical exercise (actually a set problem in Hamermesh) to obtain

$$\chi^{[2n-2,2]}=\frac{(\alpha-1)(\alpha-2)}{2}+\beta-1=\frac{\alpha(\alpha-3)}{2}+\beta$$

$$\chi^{[2n-3,3]}=\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{3!}+(\alpha-1)(\beta-1)+\gamma=\frac{\alpha(\alpha-1)(\alpha-5)}{3!}+\beta(\alpha-1)+\gamma$$

$$\chi^{[2n-4,4]}=\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-7)}{4!}+\frac{\beta(\beta-1)}{2}+\frac{\alpha(\alpha-3)}{2}\beta+(\alpha-1)\gamma+\delta$$

$$\chi^{[2n-5,5]}=\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-9)}{5!}+(\alpha-1)\left(\frac{\beta(\beta-1)}{2}\right)+\frac{\alpha(\alpha-1)(\alpha-5)\beta}{3!}$$
$$+\frac{\alpha(\alpha-3)}{2}\gamma+\beta\gamma+(\alpha-1)\delta+\varepsilon \tag{33}$$

and so on. On the other hand the characters in the Potts representation are

$$\chi^{\text{Potts}}_{(\alpha,\beta,\gamma,\delta\ldots)}=q^{n-\Sigma}(q-2)^\beta(q-3)^\gamma(q^2-4q+2)^\delta(\ldots) \tag{33a}$$

where

$$\Sigma=\beta+\gamma+2\delta+\ldots$$

and the subsequent factors may be obtained by induction using the first few (which we calculated explicitly). Comparing equations (30), (33) and (33a) we see that the coefficient of q^n in n_μ is

$$C_{n,\mu} = \delta_{n,\mu}$$

and that the coefficient of q^{n-1} is

$$C_{n-1,\mu} = \prod_{r=\mu-n}^{r=2\mu-2n} \frac{(r-(2\mu-1))}{(\mu-n+1)!}$$

provided we interpret

$$\prod_{r=a}^{r=a-b} \equiv \delta_{b,1}$$

for b strictly positive, and restrict to meaningful values of the indices. Subject to the same conditions the coefficient of q^j may be written

$$C_{j,\mu} = \left(\prod_{r=\mu-j-1}^{r=2(\mu-j-1)} [r-(2\mu-1)] \right) [(\mu-j)!]^{-1}.$$

The reader may find it instructive to work through a particular case, such as $n=2$, explicitly. The necessary tools are ready to hand in equations (30), (33) and (33a). We leave this as an exercise.

Overall we find (provided q is an integer and not Beraha) that

$$n_\mu = n_m = \prod_{r=1}^{m} \left(q - 4\cos^2\left(\frac{\pi r}{2m+a}\right)\right) \tag{34}$$

where $a=1$ if l is odd, $a=2$ if l is even and m is also the classification of irreducible representations given in I (i.e. corresponding to the representation ($s_1 = 1$, $s_{l+2} = 2m+a$)).

This result gives the degeneracies of the Potts model transfer matrix spectrum. We note in particular that the free energy, coming from $m=0$ (see Martin 1986), is unique and that the zero-temperature ground state (one eigenvalue from the $m=0$ representation and one from each of the $(q-1)$ $m=1$ representations, see Martin (1987b)) is q-fold degenerate as required.

Considering the Temperley-Lieb (or staggered ice model) representation (Temperley and Lieb 1971, Baxter 1982) of the $\{U_i\}$ it is again easy to generalise to l operators (cf Martin 1986). Here we find that

$$\chi_{\mathrm{TL}}(\tilde{t}_{l_1}\tilde{t}_{l_2}\ldots\tilde{t}_{l_i}) = 2^{l+1-j} \tag{35}$$

for

$$l_1 < l_2 < \ldots < l_i \leq l$$

so that

$$n_\mu = 2\mu + a. \tag{36}$$

This confirms the obvious generalisation of Baxter's (1986) results for $l=1$, 3 and 5.

As we have said, the above procedure does not work for the Beraha q values. For example it does not work for $q=1,2,3$. One indication of this is the different form of the 'seed' condition on similarity transformations (equation (29)) for $q=1$. For $q=2,3$ the condition cannot be generalised to larger l. As far as the Potts representation is concerned, however, it is still possible to see how the decomposition to irreducible representations proceeds by using the decompositions of the $(r-1)$-state condition relaxed μ representations discussed in I and above, and the continuity of these representations with non-Beraha-valued representations at $r \to r+\varepsilon$.

For odd l our procedure formally gives degeneracies

$$\begin{aligned} n_\mu &= -(n_{\mu-1}+n_{\mu-2}) \qquad \text{with} \qquad n_0=1,\ n_1=0 \qquad \text{for } q=1 \\ n_\mu &= (-1)^{[\mu/2]} \qquad \text{for } q=2 \\ n_\mu &= 2\sin\left(\frac{\pi}{6}+\frac{\mu\pi}{3}\right) \qquad \text{for } q=3. \end{aligned} \tag{37}$$

However, the corresponding representations are in general reducible and all the apparently negative degeneracy contributions can cancel with elements in the decomposition of positive degeneracy representations.

For example, with $l=9$ we have, for $q=1$, the following.

Representation μ	n_μ	Decomposition		
0	1	1	+41	
1	0	90		
2	−1	34+41		
3	1	$\bar{1}$+34		
4	0	9		
5	−1			$\bar{1}$

After cancelling off equivalent representations (see the examples below equation (7)) with equal and opposite 'degeneracies' we find just the one-dimensional ($s_1=1$, $s_{l+2}=1$)-state restriction observed representation remaining. Of course this is the general result for $q=1$.

For $q=2$ we have

μ	n_μ	Decomposition				
0	1	16+26				
1	1	$\bar{16}$		+74		
2	−1			74+1		
3	−1		26			+9
4	1					9
5	1				1	

and for $q=3$:

μ	n_μ	Decomposition			
0	1	41+1			
1	2	81		+9	
2	1	40			+35
3	−1				35
4	−2			9	
5	−1		1		

In general, for $q=2$, we have just one copy of each of the irreducible ($s_1=1$, $s_{l+2}=1$) and ($s_1=1$, $s_{l+2}=3$)-state restriction observed representations remaining, and for $q=3$

we have one copy of each of the irreducible $(s_1=1, s_{l+2}=1)$ and $(s_1=1, s_{l+2}=5)$ representations and two copies of $(s_1=1, s_{l+2}=3)$.

The surviving $q=2$ representations are those with vanishing idempotent (equation (7) and Temperley (1986)). If we replace $\{U_i\}$ with $\{V_i\}$ using

$$U_i=\frac{\sqrt{q}}{2}(1+V_i) \tag{38}$$

the transfer matrix (2) is then in the form given in Schultz *et al* (1964). Equation (7) becomes

$$\{V_i, V_{i+1}\}=0 \tag{39}$$

which implies that the model may be solved by a Jordan-Wigner transformation as described in that paper. The two representations are associated with different boundary conditions (see Martin 1986, Schultz *et al* 1964). This leaves $q=3$ as the simplest unsolved case.

Even this formal procedure fails for the Beraha-valued Temperley-Lieb representations. We know from explicit calculations (Martin 1986) that the decomposition here includes representations such as $(s_1=2, s_5=4)$ for $q=2$, which never occur in the Potts model and have non-vanishing idempotents.

Note that the matrix trace immediately satisfies Jones' trace condition for the Potts representation because of the degeneracies n_μ (cf Jones 1983). For example, with $\operatorname{tr}(1)\equiv 1$ and $\operatorname{Tr}(e_i^m)$ the unnormalised trace of e_j (equation (8)) in the mth irreducible representation we have, for $l=2n-1$,

$$\begin{aligned} tr\,(e_i^{\text{Potts}}) &= \left(\sum_m n_m \operatorname{Tr}(e_i^m)\right)q^{-n} \\ &= \left(\sum_m n_m {}^mC^r_{n-1}\right)q^{-n}=\frac{1}{q} \end{aligned} \tag{40}$$

(see the appendix or I). For $q=1, 2, 3$ the sum is over only the first 1, 2 or 3 irreducible representations.

To summarise: we have generalised the ABF-based scheme to construct all known representations of the $\{U_i, i=1, l\}$; we have proved that this is the complete set for $l=1, 2$ and we have obtained the degeneracies of the transfer matrix spectrum for the Potts model and the non-Beraha staggered ice model.

Appendix

For completeness we give some useful results not explicitly stated in the text.

In addition to the irreducible representations of the Temperley-Lieb algebra with $(2m+a)<r$ (m, a, r defined as in the text) there are, in general, irreducible representations associated with each $m=0, 1, 2, \ldots, n$ (where $2n$ (even) or $2n-1$ (odd) is the number of operators). A general recursion relation for the dimensions ${}^mC^r_n$ of irreducible representations may be given as follows.

For $(2m+a)<r$ (suppressing the index m)

$$C^r_n=\sum_{j=1}^{r-2} C^r_{n,j}$$

where

$$C^r_{n,j} = C^r_{n-1} - \sum_{K=1}^{j-2} C^r_{n-1,K} \qquad \text{if } (2n+a) \geqslant j+2$$

$$= 0 \quad \text{otherwise}$$

so that subsequent dimensions can be obtained from the first non-zero C^r_n for each m:

$$m=0 \qquad C^r_0 = 1 \qquad (r \geqslant 1+a)$$

$$m=1 \qquad C^r_1 = 1 \qquad (r \geqslant 3+a)$$

$$m=2 \qquad C^r_2 = 1 \qquad (r \geqslant 5+a)$$

etc

and for $kr = (2m+a)$ (where $k = 1, 2, 3, \ldots$)

$$C^r_n = C^\infty_n.$$

The proof of the recursion is indicated in I, but it should be noted (by comparison with the present result) that in I both the recursion, as it appears in the text above (6), and the initial values are wrongly stated!

An alternative description of the irreducible structure of the algebra is obtained by noting that, on discarding the last operator, the mth representation for l operators contains c copies of the $(m+1-a)$th and d copies of the $(m+2-a)$th representations for $(l-1)$ operators, where $a=1$ if $(l-1)$ is odd, $a=2$ if $(l-1)$ is even; and $c=d=0$ if the representation is not defined; $d=0$ if $(2m+4-a)=kr$ $(k=1,2,3,\ldots)$; $d=\delta$ if $(2m+4-a)=kr+1$; and $c=d=1$ otherwise (cf Jones 1983, δ determined above).

Finally we note that, provided $r > 2m+a$, the idempotents defined in (6) of the present paper may be used to give the precise form of the unnormalised idempotents R_m discussed in I, for which

$$R_m \chi R_m = \xi_m(\chi) R_m$$

with χ any product of U_i operators and $\xi_m(\chi)$ a scalar. The precise form is

$$R_m = \left(\prod_{i=1}^{n-m} U_{2i-1} \right) \mathrm{idem}_{n-m}[m+1]$$

where $\mathrm{idem}_{n-m}[m+1]$ is obtained from $\mathrm{idem}[m+1]$ by making the substitution

$$U_i \to U_{i+2(n-m)}$$

(the form given in I is correct only for $m=0, 1$).

References

Andrews G E, Baxter R J and Forrester P J 1984 *J. Stat. Phys.* **35** 193
Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
—— 1986 lecture, unpublished
Birman J S 1974 *Braids, Links and Mapping Class Groups, Ann. Math. Studies* vol 82 (Princeton, NJ: Princeton University Press)
Friedan D, Qiu Z and Shenker S 1983 *Preprint, Conformal Invariance, Unitarity and two-dimensional critical exponents, Workshop on Vertex Operators, MSRI Berkeley*
Hamermesh M 1962 *Group Theory* (Oxford: Pergamon)
Jones V F R 1983 *Invent. Math.* **72** 1

—— 1985 *Bull. Am. Math. Soc.* **12** 103
Kauffman L H 1986a *Preprint, New Invariants in the Theory of Knots*
—— 1986b *Preprint, Statistical Mechanics and the Jones polynomial*
Kuniba A, Akutsu Y and Wadati M 1986 *J. Phys. Soc. Japan* **55** 3285
Leech J W and Newman D J 1970 *How to use Groups* (London: Methuen)
Magnus W, Karrass A and Solitar D 1975 *Combinational Group Theory* (New York: Dover)
Martin P P 1986 *J. Phys. A: Math. Gen.* **19** L1117
—— 1987a *J. Phys. A: Math. Gen.* **20** L539
—— 1987b *J. Phys. A: Math. Gen.* **20** L601
Schultz T D, Mattis D C and Lieb E 1964 *Rev. Mod. Phys.* **36** 856
Temperley H N V 1986 *Preprint, Potts models and related problems*
Temperley H N V and Lieb E H 1971 *Proc. R. Soc.* A **322** 251
Tung W K 1985 *Group Theory in Physics* (Singapore: World Scientific)